USING OF THE TRIGONOMETRIC SERIES IN THE APPROXIMATING FLOWS OF VISCOUS HEAT CONDUCTING GAS

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The complete system of Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \delta}{\partial t} + \mathbf{V} \cdot \nabla \delta - \delta \, \mathrm{div} \mathbf{V} = 0, \\ \dfrac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} + \dfrac{1}{\gamma} \delta \nabla p = \mu_0 \delta \left[\dfrac{1}{4} \nabla \left(\mathrm{div} \mathbf{V} \right) + \dfrac{3}{4} \Delta \mathbf{V} \right], \\ \dfrac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \gamma p \, \mathrm{div} \mathbf{V} = \varkappa_0 \Delta (\delta p) + \Phi \left(\mu_0, \mathbf{V} \right). \end{cases}$$

$$\Phi\left(\mu_0, \mathbf{V}\right) = \mu_0 \gamma (\gamma - 1) \left\{ \frac{1}{2} \left[\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} \right)^2 + \left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_3}{\partial x_3} \right)^2 + \left(\frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3} \right)^2 \right] + \right.$$
$$\left. + \frac{3}{4} \left[\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right)^2 + \left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right)^2 + \left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right)^2 \right] \right\},$$

$\mu_0 = \mathrm{const}$, $\varkappa_0 = \mathrm{const}$, $\delta = 1/\rho$ – specific volume, $p = \rho T$, $e = T$.

Bautin S.P. The characteristic Cauchy problem and its applications in gas dynamics.Nauka, 2009.

Titov S.S.The space-periodic solutions of the complete Navier-Stokes equations //The reports of the Academy of Sciences. 365.

The one-dimensional initial-boundary value problem

$$\begin{cases} \delta_t = \delta u_x - u\delta_x, \\ u_t = -uu_x - \dfrac{1}{\gamma}\delta p_x + \mu_0 \delta u_{xx}, \\ p_t = -up_x - \gamma p u_x + \varkappa_0 (\delta p)_{xx} + \mu_0 \gamma(\gamma - 1) u_x^2, \\ \delta|_{t=0} = \delta^o(x), \\ u|_{t=0} = u^o(x), \\ p|_{t=0} = p^o(x), \\ u|_{x=0,\ x=\pi} = 0, \quad T_x|_{x=0,\ x=\pi} = 0, \quad t \geq 0, \quad 0 \leq x \leq \pi. \end{cases} \tag{1}$$

The set of the initial-boundary value problem has a unique solution in the space L_2, and under additional hypothesis in the spaces $C^{2+\alpha,1+\alpha/2}$, (x,t) too.

Antontsev S.N., Kazhikhov A.V., Monakhov V.N. The boundary value problems in the mechanics of the inhomogeneous liquids. Nauka, 1983.

Kazhikhov A.V. The selected works. Mathematical Hydrodynamics. Novosibirsk: 2008.

The type of solutions and solution construction

$$\delta(t,x) = 1 + \sum_{k=1}^{\infty} \delta_k(t)\cos kx, \quad u(t,x) = \sum_{k=1}^{\infty} u_k(t)\sin kx,$$
$$p(t,x) = 1 + p_0(t) + \sum_{k=1}^{\infty} p_k(t)\cos kx, \tag{2}$$

The initial data:

$$\delta(t,x)|_{t=0} = 1 + \sum_{k=1}^{\infty} \delta_k(0)\cos kx, \quad u(t,x)|_{t=0} = \sum_{k=1}^{\infty} u_k(0)\sin kx,$$
$$p(t,x)|_{t=0} = 1 + p_0(0) + \sum_{k=1}^{\infty} p_k(0)\cos kx, \tag{3}$$

For the representations (2) at $x = 0$ and $x = \pi$ the conditions of adhesion and thermal insulation are fulfilled.

The convergence of views (2) is proved by Titiv S.S. and Kurmaeva K. V.

The representations (2) are substituted into the system (1) and each equation is projected onto its system of basic harmonics. We obtain the infinite system of ordinary differential equations.

$$\delta_\ell'(t) = \ell u_\ell(t) + \frac{2}{\pi} \sum_{k,m=1}^{\infty} (m a_{km\ell} + k b_{km\ell})\, \delta_k(t) u_m(t); \qquad (4)$$

$$u_\ell'(t) = -\frac{2}{\pi} \sum_{k,m=1}^{\infty} m b_{k\ell m} u_k(t) u_m(t) + \frac{1}{\gamma} \ell p_\ell(t) +$$

$$+\frac{2}{\gamma\pi} \sum_{k,m=1}^{\infty} m b_{m\ell k} \delta_k(t) p_m(t) - \mu_0 \ell^2 u_\ell(t) - \qquad (5)$$

$$-\mu_0 \frac{2}{\pi} \sum_{k,m=1}^{\infty} m^2 b_{m\ell k} \delta_k(t) u_m(t);$$

$$
\begin{aligned}
p_\ell'(t) = {} & \frac{2}{\pi}\sum_{k,m=1}^{\infty}(mb_{km\ell}-\gamma k a_{km\ell})\,u_k(t)p_m(t) - \gamma[1+p_0(t)]\ell u_\ell(t) - \\
& -\varkappa_0 \ell^2\{[1+p_0(t)]\delta_\ell(t)+p_\ell(t)\} - \\
& -\varkappa_0\frac{2}{\pi}\sum_{k,m=1}^{\infty}\left[(m^2+k^2)a_{km\ell}-2kmb_{km\ell}\right]\delta_k(t)p_m(t) + \\
& +\mu_0\gamma(\gamma-1)\frac{2}{\pi}\sum_{k,m=1}^{\infty} km\,a_{km\ell}u_k(t)u_m(t);
\end{aligned}
\tag{6}
$$

$$
p_0'(t) = \frac{1}{2}(1-\gamma)\sum_{k=1}^{\infty} k u_k(t)p_k(t) + \frac{1}{2}\mu_0\gamma(\gamma-1)\sum_{k=1}^{\infty}k^2u_k^2(t); \tag{7}
$$

$$
a_{k,m,\ell} = \int_0^\pi \cos(kx)\,\cos(mx)\,\cos(\ell x)\,dx;\quad b_{k,m,\ell} = \int_0^\pi \sin(kx)\,\sin(mx)\,\cos(\ell x)\,dx.
$$

Theorems on multiple frequencies

The first theorem on multiple frequencies.

If the initial data contains the harmonic with frequency ℓ_1, then the solution constains harmonics with the respect to the spatial variable only with the harmonics $\ell_1, 2\ell_1, 3\ell_1, \ldots$

The second theorem on multiple frequencies.

If the initial data comprise the harmonic with frequencies ℓ_1, $\ell_2, \ldots, 3\ell_n$, then in the solution there are harmonics with respect to the spatial variable only with the frequencies that are multiples of d: $d, 2d, 3d, \ldots$

Where d is the greatest common divisor.

The thorems are proved for the infinite system (4) - (7).

Zamyslov V.E. The standing waves as the solution of the complete Navier-Stokes equations in the one-dimensional case // Computational technologies. 2013. T. 18.

The theorems are reproved by Titov S.S., Kurmaeva K.V.

The redistribution of the initial perturbation for the other harmonics

$$\ell_1 = d = 5$$

Appearance the frequency

$$\ell_2 = 10, \ell_2 = 15, \ell_3 = 20, ...$$

The theorem on the multiple frequencies in the three-dimensional case

The type of solution

$$\delta(t,x_1,x_2,x_3) = 1 + \delta_0(t) + \delta_1(t,x_1) + \delta_2(t,x_2) + \delta_3(t,x_3),$$

$$u(t,x_1,x_2,x_3) = u_1(t,x_1) + u_2(t,x_2) + u_3(t,x_3),$$

$$v(t,x_1,x_2,x_3) = v_1(t,x_1) + v_2(t,x_2) + v_3(t,x_3),$$

$$w(t,x_1,x_2,x_3) = w_1(t,x_1) + w_2(t,x_2) + w_3(t,x_3),$$

$$p(t,x_1,x_2,x_3) = 1 + p_0(t) + p_1(t,x_1) + p_2(t,x_2) + p_3(t,x_3)$$

Pogodin Yu.A., Suchkov V.A., Yanenko N.N. The travelling waves in gas dynamics equations // Dokl. T. CXIX, No. 4. 1957.

Sidorov A.F., Shapeev V.P., Yanenko N.N. The method of the differential relations and its applications in gas dynamics. Nauka, 1984.

The solution is constructed in the following form:

$$\begin{cases} \delta(t,x_1,x_2,x_3) = 1+\delta_0(t)+\sum\limits_{j=1}^{3}\delta_j(t,x_j), \\ u(t,x_1,x_2,x_3) = \sum\limits_{j=1}^{3} u_j(t,x_j), \\ v(t,x_1,x_2,x_3) = \sum\limits_{j=1}^{3} v_j(t,x_j), \\ w(t,x_1,x_2,x_3) = \sum\limits_{j=1}^{3} w_j(t,x_j), \\ p(t,x_1,x_2,x_3) = 1+p_0(t)+\sum\limits_{j=1}^{3} p_j(t,x_j) \end{cases} \tag{2}$$

$$f_j(t,x_j)=\sum_{k=1}^{\infty}\left[f_{j,k,1}(t)\cos(kx_j)+f_{j,k,2}(t)\sin(kx_j)\right];$$

$f: \delta, u, v, w, p;$ the first index $j = 1, 2, 3$ – the number of the spatial variable;
the second index – the harmonic frequency.

The result of the projecting the first equation of the system (1)

$$\delta_0'(t) = \frac{1}{2}\sum_{k=1}^{\infty} k\Big\{ - u_{1k1}(t)\delta_{1k2}(t) + u_{1k2}(t)\delta_{1k1}(t) -$$

$$-v_{2k1}(t)\delta_{2k2}(t) + v_{2k2}(t)\delta_{2k1}(t) - w_{3k1}(t)\delta_{3k2}(t) + w_{3k2}(t)\delta_{3k1}(t) +$$

$$+\delta_{1k1}(t)u_{1k2}(t) - \delta_{1k2}(t)u_{1k1}(t)\delta_{1k1}(t) + \delta_{2k1}(t)v_{2k2}(t) - \delta_{2k2}(t)v_{2k1}(t) +$$

$$+\delta_{3k1}(t)w_{3k2}(t) - \delta_{3k2}(t)w_{3k1}(t)\Big\};$$

$$\delta_{1\ell 1}'(t) = [1+\delta_0(t)]\ell u_{1\ell 2}(t) + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m\Big[- a_{km\ell}u_{1k1}(t)\delta_{1m2}(t) +$$

$$+b_{km\ell}u_{1k2}(t)\delta_{1m1}(t) + a_{km\ell}\delta_{1k1}(t)u_{1m2}(t) - b_{km\ell}\delta_{1k2}(t)u_{1m1}(t)\Big];$$

$$\delta_{1\ell 2}'(t) = -[1+\delta_0(t)]\ell u_{1\ell 1}(t) + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m\Big[b_{m\ell k}u_{1k1}(t)\delta_{1m1}(t) -$$

$$-b_{k\ell m}u_{1k2}(t)\delta_{1m2}(t) - b_{m\ell k}\delta_{1k1}(t)u_{1m1}(t) + b_{k\ell m}\delta_{1k2}(t)u_{1m2}(t)\Big];$$

$$\delta'_{2\ell 1}(t) = [1+\delta_0(t)]\ell v_{2\ell 2}(t) + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m\Big[-a_{km\ell}v_{2k1}(t)\delta_{2m2}(t)+$$
$$+b_{km\ell}v_{2k2}(t)\delta_{2m1}(t) + a_{km\ell}\delta_{2k1}(t)v_{2m2}(t) - b_{km\ell}\delta_{2k2}(t)v_{2m1}(t)\Big];$$
$$\delta'_{2\ell 2}(t) = -[1+\delta_0(t)]\ell v_{2\ell 1}(t) + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m\Big[b_{m\ell k}v_{2k1}(t)\delta_{2m1}(t)-$$
$$-b_{k\ell m}v_{2k1}(t)\delta_{2m1}(t) - b_{m\ell k}\delta_{2k1}(t)v_{2m1}(t) + b_{k\ell m}\delta_{2k2}(t)v_{2m2}(t)\Big];$$

$$\delta'_{3\ell 1}(t) = [1+\delta_0(t)]\ell w_{3\ell 2}(t) + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m\Big[-a_{km\ell}w_{3k1}(t)\delta_{3m2}(t)+$$
$$+b_{km\ell}w_{3k2}(t)\delta_{3m1}(t) + a_{km\ell}\delta_{3k1}(t)w_{3m2}(t) - b_{km\ell}\delta_{3k2}(t)w_{3m1}(t)\Big];$$
$$\delta'_{3\ell 2}(t) = -[1+\delta_0(t)]\ell w_{3\ell 1}(t) + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m\Big[b_{m\ell k}w_{3k1}(t)\delta_{3m1}(t)-$$
$$-b_{k\ell m}w_{3k2}(t)\delta_{3m2}(t) - b_{m\ell k}\delta_{3k1}(t)w_{3m1}(t) + b_{k\ell m}\delta_{3k2}(t)w_{3m2}(t)\Big].$$

The theorem on the multiple frequencies. *Imagine you are given three sets of positive integers and their greatest divisors:*

$$d_1 = НОД(\ell_0, ..., \ell_L); \; d_2 = НОД(m_0, ..., m_M); \; d_3 = НОД(n_0, ..., n_N).$$

Let at $t = 0$ the initial data contain a finite number of harmonics. Moreover, the harmonics that depend on x_1 have a frequency ℓ_0, ...,ℓ_L; dependent on x_2 – the frequency of the m_0, ..., m_M; depend on x_3 – frequency of the n_0, ..., n_N. Then the solution of the infinite system of the ordinary differential equations, all coefficients $f_{i\ell j}(t)$ for which the second index ℓ is not a multiple of d_i, will be identical to zeros.

The theorem says that the solution presented in (2) and containing at the initial moment of time only a finite number of harmonics, for $t > 0$ can be harmonics from x_i only with the frequencies that are multiples of $d_i, i = 1, 2, 3$.

The frequency division in the following areas: dividing in the initial conditions of the spatial variables received that the decision of the frequencies of the harmonics were also divided with respect to spatial variables.

The meaning of the theorem

Remark 1. The theorem is valid for the system of gas dynamics equations, which is obtained from the system (1), if it is put to the $\mu_0 = \varkappa_0 = 0$.

Remark 2. From the theorem it follows that the solutions of the form (2) do not have the property of «doubling the frequency», which is sometimes suggested in the viscous continuum.
Landau L.D., Lifshitz E. M. Theoretical physics. Volume VI. Hydrodynamics. Nauka, 1988.

Remark 3. As a natural phenomenon the well-known fact is that the excitation of the harmonics with the certain frequencies over time are fixed and the new harmonics, the frequencies of which are fully consistent with this theorem and the similar theorems in the one-dimensional case.
Aldoshina I., Pritts R. Musical acoustics. «Composer», 2006.

Remark 4. The authors of the theorem presented here do not know the papers in which, for at least for formal solutions of specific nonlinear partial differential equations the fact corresponding to theorems on multiple frequencies would be proved.

The mapping if $\mu_0 \neq 0, \varkappa_0 \neq 0$ with case $\mu_0 = 0, \varkappa_0 = 0$

$\mu_0 \neq 0, \varkappa_0 \neq 0$

$\mu_0 = 0, \varkappa_0 = 0$

Conclusion

1. Using infinite trigonometric series we constructed the non-stationary solutions of the comlete system of Navier–Stokes equations approximately.

2. The theorems on the multiple frequencies have been proved and thus the fact the fact of the redistribution of the initial perturbations from one harmonic to another has been establisheal and the mathematical substantiation of the rules for the formation of overtones is given.

3. The algebraic structure of the set of frequencies of the harmonics arising in the solution is established depending on the frequencies of the harmonics including in the initial conditions.

4. The corresponding computational methods confirmed that the approximate solutions constructed in the case of relatively small initial data transfer the one-dimensional flows of the compressible viscous heat-conducting gas with enough accuracy.

5. The examples of sufficiently nontrivial solutions are given.

6. The proposed approach make it possible to obtain the solutions at sufficiently large intervals of time, including when the flow is practically stabilized in this case to homogeneous quiescence rest. The calculations have shown that the time stabilization of the flow – about $1/\mu_0$.

Thank you for your attention !

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