FOCUSING CYLINDRICALLY SYMMETRIC SHOCK WAVE IN A GAS

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Formulation of the problem

We set four parameters with different dimensions:

$U_{g0}, r_0, t_0, \rho_0.$

Initial gas parameters: $\rho_0 = const, U_0 = 0, P_0 = 0, E_0 = 0.$

On the cylinder boundary at the point t_0, r_0 is set velocity $U_{g0} < 0$.

Conservation laws on the shock wave:

$$\rho_w \left(D - U_w \right) - \rho_0 D = 0, \tag{1}$$

$$\rho_0 D U_w - P_w = 0, \tag{2}$$

$$\rho_0 D \left(E_w + \frac{1}{2} U_w^2 \right) - P_w U_w = 0. \tag{3}$$

Equations (1)–(3) are closed by the equation of state

$$P = \left(\gamma - 1 \right) \rho E, \; P = F\left(S \right) \rho^{\gamma}. \tag{4}$$

$$M_w = \pi \rho_0 r_w^2,$$ - Lagrangian coordinate of shock.

$$W = \frac{dM_w}{dt} = 2\pi \rho_0 r_w D.$$ - shock wave velocity in Lagrangian coordinates

With respect to this expressions from (1) - (4) follow:

$$\rho_w = \frac{\gamma+1}{\gamma-1}\rho_0,$$

$$P_w = \frac{2}{\gamma+1}\rho_0^{1/3}\left(4\pi\right)^{-2/3}\left(3M_w\right)^{-4/3}W^2,$$

$$U_w = \frac{2}{\gamma+1}\left(4\pi\rho_0\right)^{-1/3}\left(3M_w\right)^{-2/3}W,$$

$$F_w = \left[\frac{2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\rho_0^{-(\gamma-1/3)}\left(4\pi\right)^{-2/3}\right]\left(3M_w\right)^{-4/3}W^2.$$

We take the equation of shock trajectory in the form:

$$M_w = M_0 \left(\frac{t_f - t}{t_f - t_0} \right)^n , \tag{5}$$

where t_f - focusing time.

$$W = W_0 \left(\frac{t_f - t}{t_f - t_0} \right)^{n-1} ,$$

where $W_0 = \left(2M_0 \right)^{1/2} \left(2\pi\rho_o \right)^{1/2} \frac{\left(\gamma + 1 \right)}{2} U_{g0},$

$$t_f = t_0 - \frac{M_0 n}{W_0}.$$

Gas flow behind the shock wave

Adiabatic flow parameters behind the shock wave are defined by equations of trajectory, conservation of mass and motion:

$$\left(\frac{\partial r}{\partial t}\right)_M - U = 0, \tag{6}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_M + 2\pi\rho^2 \frac{\partial (rU)}{\partial M} = 0, \tag{7}$$

$$\left(\frac{\partial U}{\partial t}\right)_M + 2\pi r \frac{\partial (F\rho^\gamma)}{\partial M} = 0. \tag{8}$$

Change in (6)-(8) to the new unknown functions

$$R = r^2, \quad C = rU.$$

We get:

$$\left(\frac{\partial R}{\partial t}\right)_M - 2C = 0, \tag{9}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_M + 2\pi\rho^2 \frac{\partial C}{\partial M} = 0, \tag{10}$$

$$\left(\frac{\partial C}{\partial t}\right)_M + 2\pi R \frac{\partial\left(F\rho^\gamma\right)}{\partial M} - C^2 R^{-1} = 0. \tag{11}$$

New functions on the shock wave have the form

$$R_w = R_0 \frac{M_w}{M_0}, \quad C_w = C_0 \left(\frac{M_w}{M_0}\right)^{(n-1)/n}. \tag{12}$$

Change from variables t, M to variables t, $\xi(t,M)$.
We set the dependence $\xi(t,M)$ in form:

$$\xi = \frac{M}{M_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{-n}.$$

Representing R, ρ and C as the product of the functions depending on time and the functions of ξ:

$$R = \alpha_R(t)T(\xi) \quad \rho = \alpha_\rho(t)\delta(\xi) \quad C = \alpha_C(t)Z(\xi)$$

Determine this dependences on shock wave, i.e. at $\xi=1$

$$T_w = T_1, \quad \alpha_R(t) = R_0\left(\frac{t_f - t}{t_f - t_0}\right)^n T_1^{-1},$$

$$\delta_w = \delta_1, \quad \alpha_\rho = \rho_0\left(\frac{\gamma+1}{\gamma-1}\right)\delta_1^{-1}, \tag{13}$$

$$Z_w = Z_1, \quad \alpha_C(t) = C_0\left(\frac{t_f - t}{t_f - t_0}\right)^{n-1} Z_1^{-1}.$$

So system of equations (9)-(11) transform to the system of equations for T, δ и Z:

$$\xi T' = A_1, \qquad (14)$$

$$\delta_1 B_1 Z' - \xi Z_1 \delta' = 0, \qquad (15)$$

$$-\frac{\xi}{Z_1} Z' + \frac{C_1 \gamma \xi}{\delta_1} \delta' = C_2. \qquad (16)$$

where

$$A_1 = T - \frac{2Z T_1}{(\gamma+1) Z_1}, \quad B_1 = \frac{2\delta^2}{(\gamma-1)\delta_1^2}, \quad C_1 = \frac{\delta^{\gamma-1} T \xi^{-2/n}}{\delta_1^{\gamma-1} T_1},$$

$$C_2 = \frac{Z^2 T_1}{(\gamma+1) Z_1^2 T} - \frac{(n-1)Z}{n Z_1} - C_1 \frac{(n-2)\delta}{n \delta_1}.$$

Equations (14) - (16) are form the system of linear inhomogeneous equations regarding to the T', δ', Z'.

The system determinant is

$$\Delta = B_1 C_1 \gamma \xi - \xi^2.$$

At $\Delta \neq 0$ unique solution of the system (14)–(16) exists. It has the form

$$T' = \frac{A_1}{\xi}, \quad \delta' = \frac{B_1 C_2 \delta_1}{\Delta}, \quad Z' = \frac{\xi C_2 Z_1}{\Delta}. \tag{17}$$

The equations (17) are numerically integrated from the shock wave, where ξ=1, to value ξ_*, at which $\Delta(\xi_*)$=0. Value n_* is determined such that simultaneously $\Delta(\xi_*)$=0 and $C_2(\xi_*)$=0.

Values ξ_* and n_* depend on γ. They are given in the table

γ	1.1	1.2	4/3	1.4	5/3
n_*	1.770501	1.722331	1.684516	1.670651	1.631252
ξ_*	6.768540	4.873946	3.896265	3.616019	2.990161

Solution at the exceptional point

At the exceptional point determinant of the system (14)-(16) and C_2 vanish simultaneously

$$\frac{2\gamma}{\gamma-1}\frac{T_* \, \delta_*^{\gamma+1} \, \xi_*^{(n-2)/n}}{T_1 \, \delta_1^{\gamma+1}} - \xi_*^2 = 0. \tag{18}$$

$$\frac{1}{\gamma+1}\left(\frac{Z_*}{Z_1}\right)^2 - \frac{n-1}{n}\left(\frac{Z_*}{Z_1}\right)\left(\frac{T_*}{T_1}\right) - \frac{2(\gamma-1)(n-2)}{n\gamma(\gamma+1)}\left(\frac{T_*}{T_1}\right)\xi_* = 0. \tag{19}$$

From the expression of Δ'

$$\Delta' = \left(\Delta + \xi^2\right)\left(\frac{2(\gamma+1)\delta C_2}{(\gamma-1)\delta_1 \Delta} - \frac{4Z\,T_1}{(\gamma+1)Z_1\,T\xi} + \frac{2(n-1)}{n\,\xi}\right) - 2\xi.$$

follows, that $\Delta'=\infty$ at $\Delta=0$.

We interpolate $\Delta(\xi)$ behavior by a function of ξ to determine value ξ_*

$$\Delta = \frac{1}{a}\sqrt{\xi\left(\xi_* - \xi\right)}. \tag{20}$$

Derivative Δ' has the form $\Delta' = \frac{1}{2a^2\Delta}\left(\xi_* - 2\xi\right).$

Constant value a in (20) is defined at the point ξ_ν closest to ξ_* and such that $\Delta(\xi_\nu)$ and $\Delta'(\xi_\nu)$ are still finite. Then

$$\xi_* = \frac{2\xi_\nu\left(\Delta_\nu - \xi_\nu\Delta'_\nu\right)}{\Delta_\nu - 2\xi_\nu\Delta'_\nu}.$$

After that we found T_* by extrapolation

$$T_* = T_\nu + T'_\nu\left(\xi_* - \xi_\nu\right).$$

Value δ_* is determined from the condition $\Delta_* = 0$.

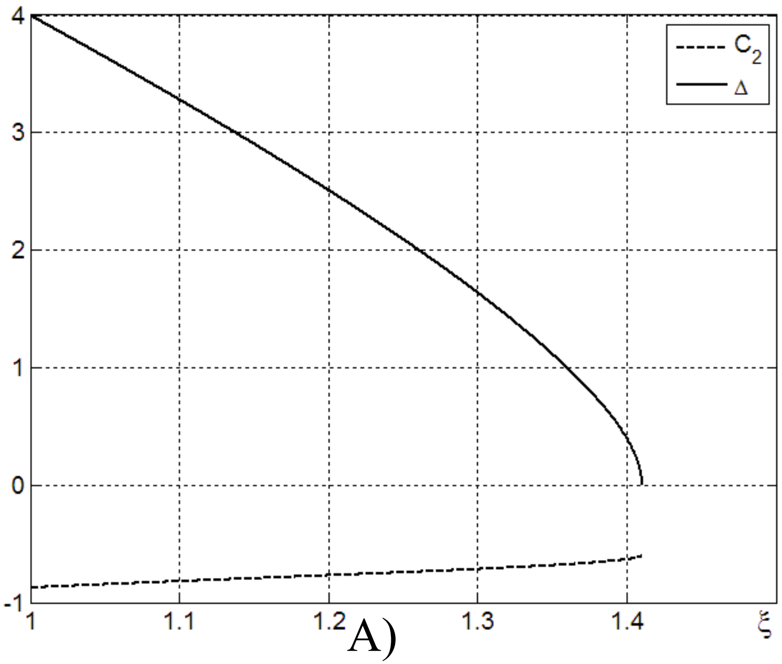
$$\delta_* = \delta_1 \left(\frac{\gamma - 1}{2\gamma} \frac{T_1}{T_*} \xi_*^{(n+2)/n} \right)^{1/(\gamma+1)},$$

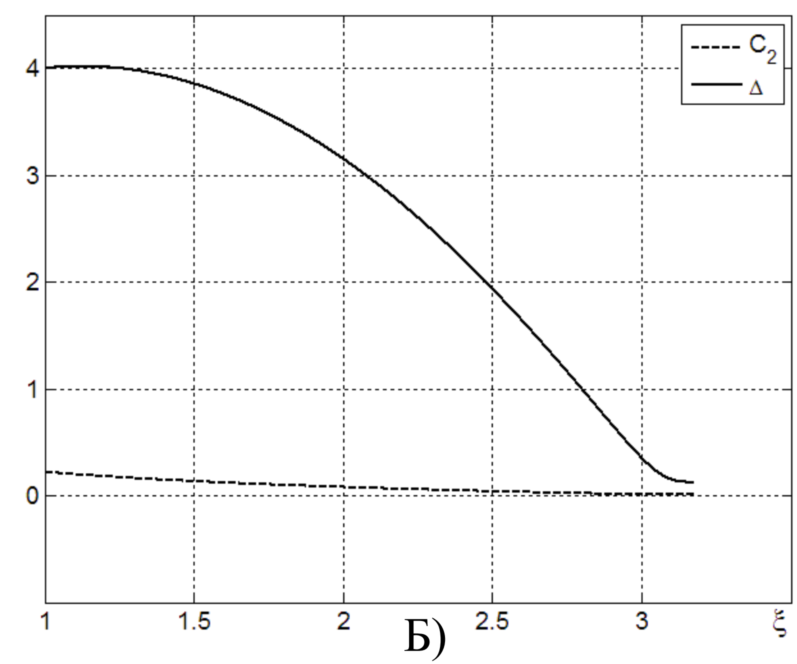
and value Z_* is found from the (19) where $C_2(\xi_*)=0$.

After evaluation of uncertainty forms at the ξ_*

$$\frac{C_2(\xi_*)}{\Delta(\xi_*)} = \frac{0}{0}$$

integration of equations (17) continues for up to $\xi \approx 10^{10}$.





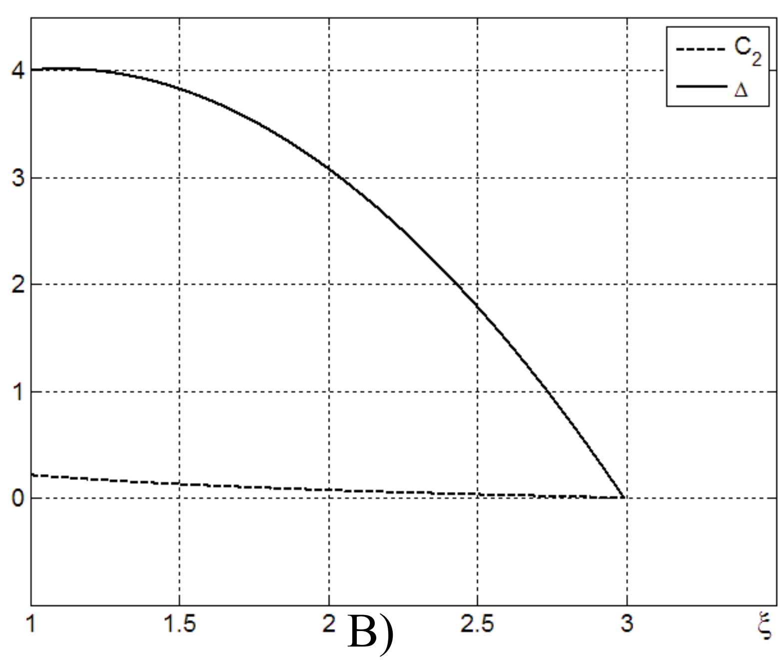
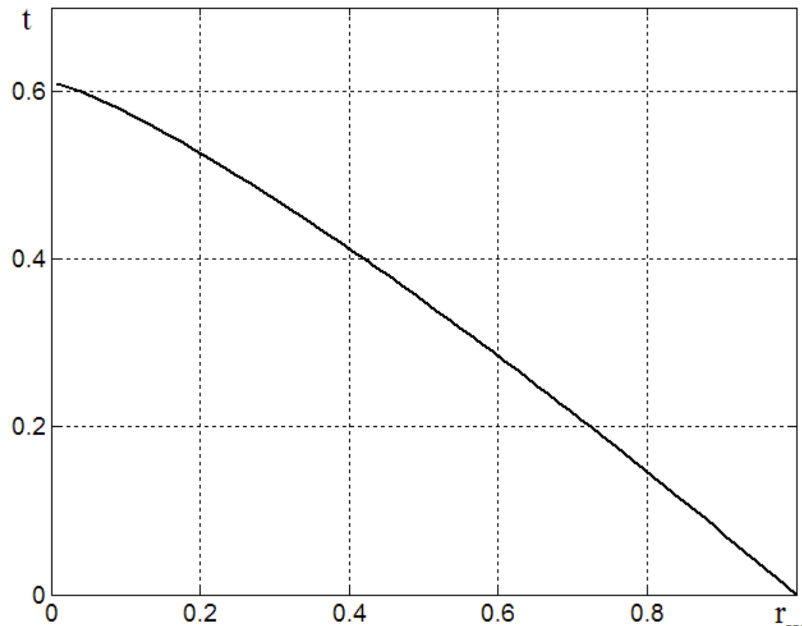


Fig. 1. Fragments of the search of n_*. The dependence of the determinant Δ and the coefficient C_2 on ξ for γ=*5/3*.

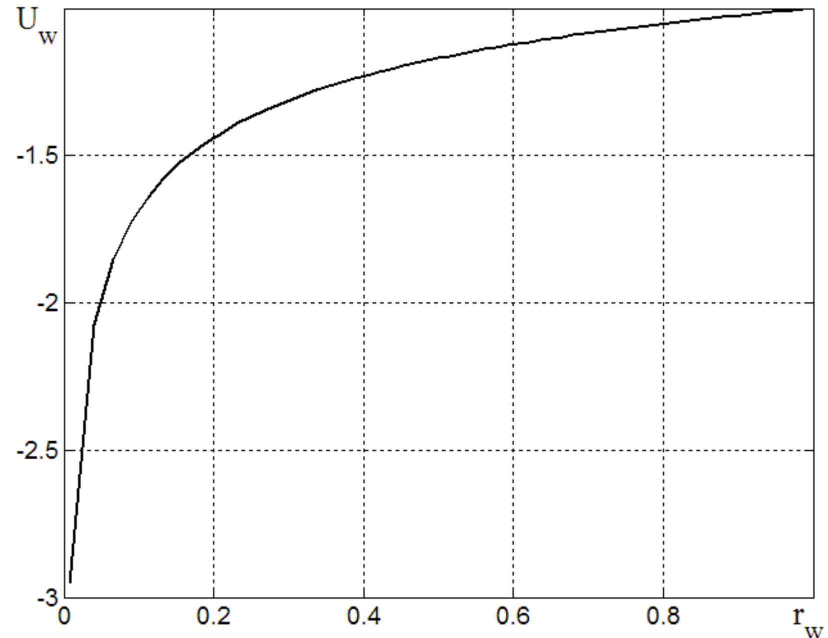
A) *n=4*

Б) *n=1,625*

В) $n = n_* = 1{,}631252$.



A)



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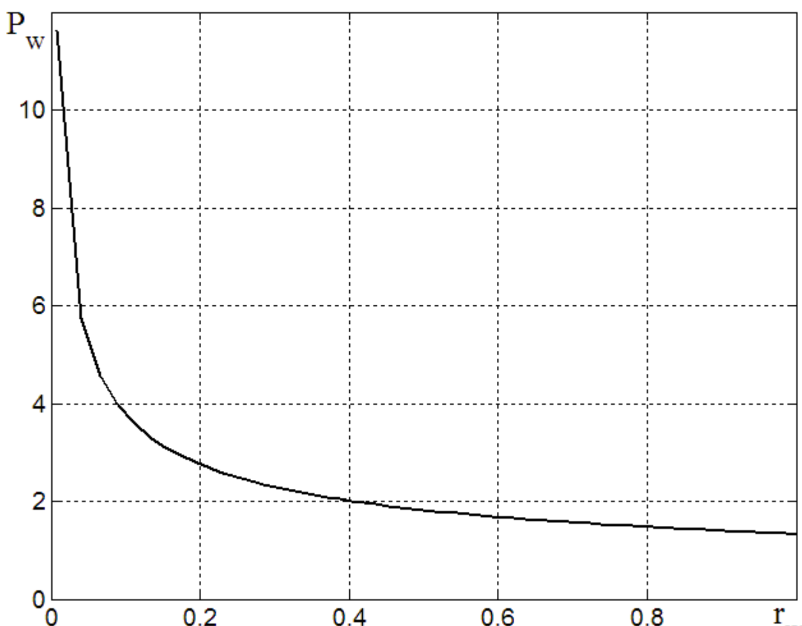
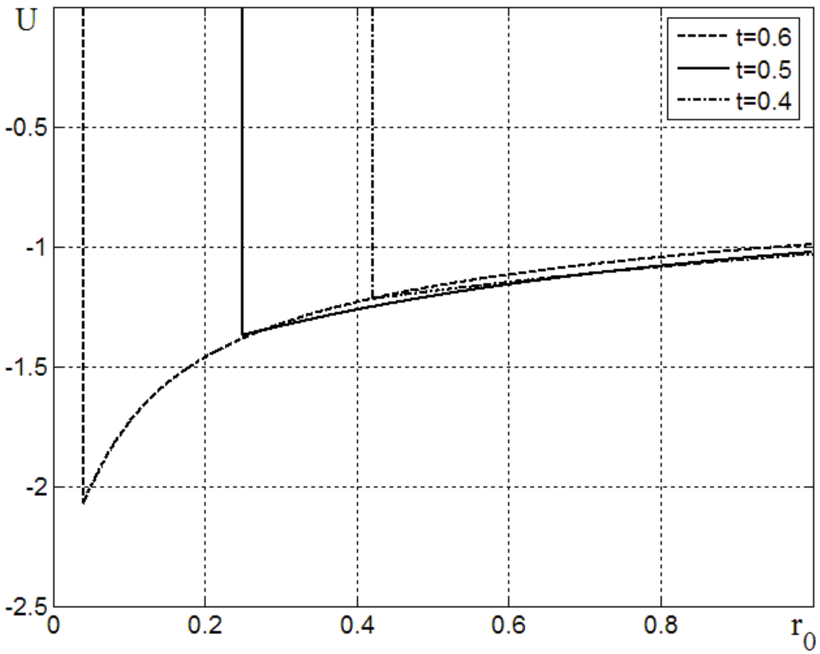
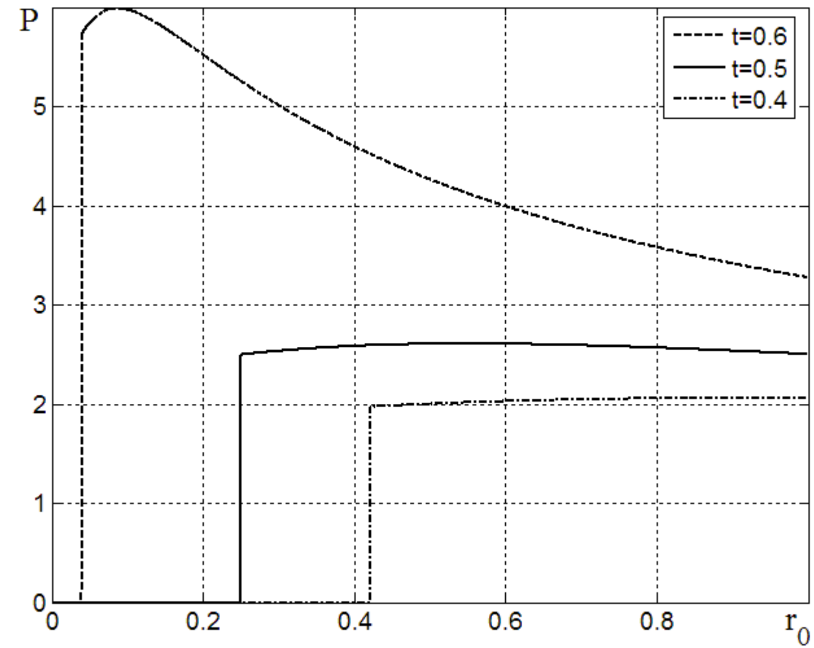
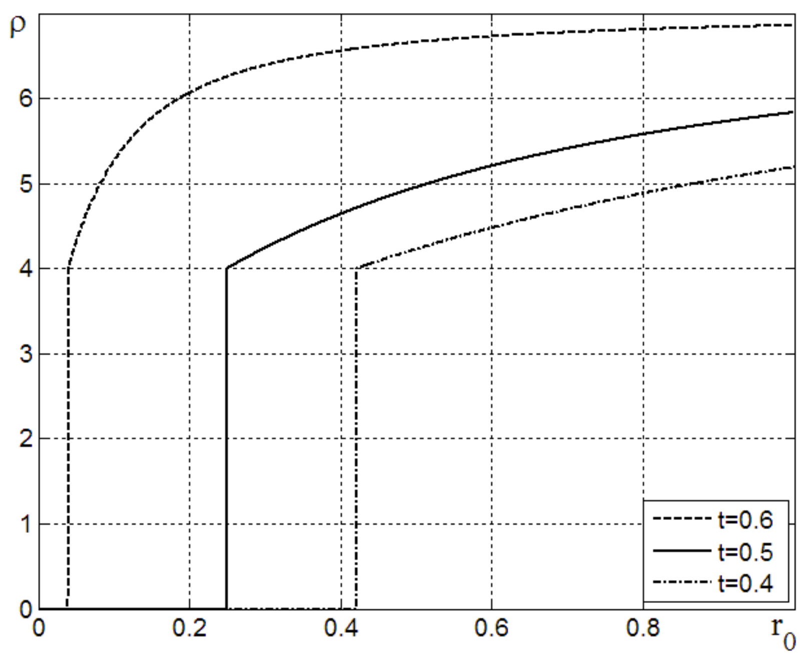


В)

Рис. 2. A) The trajectory of the shock; Б) The velocity dependence on the shock front on Euler coordinates; В) The pressure dependence on the shock front on Euler coordinates for γ=5/3 and n_*=1,631252.







ig. 3. The dependencies of the elocity, pressure and density on Euler oordinates for γ=5/3, n_*=1,631252 for me points t=0.4; 0.5; 0.6.

Thank you for attention!