CONVERGENT SHOCK IN A GAS FOR LARGE VALUES OF A SELF-SIMILAR COEFFICIENT

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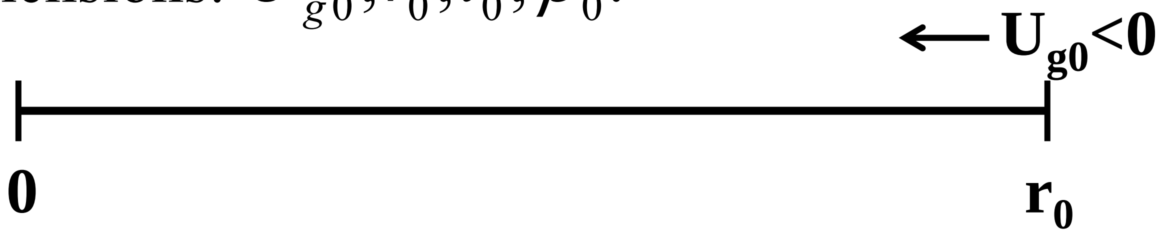
In formulating of the problem we assign four parameters with different dimensions: U_{g0}, r_0, t_0, ρ_0.



Initial parameters for the gas: $\rho_0 = const, U_0 = 0, P_0 = 0, E_0 = 0$

Conservation laws on the shock wave:

$$\rho_w\left(D - U_w\right) - \rho_0 D = 0, \tag{1}$$

$$\rho_0 D U_w - P_w = 0, \tag{2}$$

$$\rho_0 D\left(E_w + \frac{1}{2}U_w^2\right) - P_w U_w = 0. \tag{3}$$

Equations (1) - (3) are closed by the equation of state

$$P = \left(\gamma - 1\right)\rho E,\ \ P = F\left(S\right)\rho^{\gamma}. \tag{4}$$

$$M_w = \frac{4}{3}\pi\rho_0 r_w^3$$ - the Lagrangian coordinate of the shock wave

$$W = (3M_w)^{2/3}(4\pi\rho_0)^{1/3} D$$ - velocity of the shock wave in Lagrangian coordinates

Expressed D and substituting in (1) - (3), we obtain the conditions on the shock wave, which contain W and M_w

$$\left(\frac{1}{\rho_w} - \frac{1}{\rho_0}\right)W + (4\pi)^{1/3}\left(\frac{3M_w}{\rho_0}\right)^{2/3} U_w = 0, \tag{5}$$

$$U_w W - (4\pi)^{1/3}\left(\frac{3M_w}{\rho_0}\right)^{2/3} P_w = 0, \tag{6}$$

$$\left(E_w + 0,5U_w^2\right)W - (4\pi)^{1/3}\left(\frac{3M_w}{\rho_0}\right)^{2/3} P_w U_w = 0. \tag{7}$$

The expressions follow from (4), (5), (6) and (7)

$$\rho_w = \frac{\gamma+1}{\gamma-1}\rho_0,$$

$$U_w = \frac{2W}{(\gamma+1)(4\pi\rho_0)^{1/3}(3M_w)^{2/3}},$$

$$P_w = \frac{2\rho_0^{1/3}W^2}{(\gamma+1)(4\pi)^{2/3}(3M_w)^{4/3}},$$

$$F_w = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\frac{W^2\rho_0^{(1-3\gamma)/3}}{(4\pi)^{2/3}(3M_w)^{4/3}}.$$

We define the trajectory of the shock wave in the form

$$M_w = M_0 \left(\frac{t_f - t}{t_f - t_0} \right)^n , \tag{8}$$

where t_f – focusing time.

$$W = W_0 \left(\frac{t_f - t}{t_f - t_0} \right)^{n-1} ,$$

where $W_0 = 3 M_0^{2/3} \left(4\pi\rho_0 \right)^{1/3} \frac{(\gamma + 1)}{2} U_g ,$

$$t_f = t_0 - \frac{M_0 n}{W_0} .$$

Parameters of the adiabatic flow between the shock wave and boundary of the gas are determined by the equations of the trajectory, the conservation of mass and motion

$$\left(\frac{\partial r}{\partial t}\right)_M - U = 0, \tag{9}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_M + 4\pi\rho^2 \frac{\partial\left(r^2 U\right)}{\partial M} = 0, \tag{10}$$

$$\left(\frac{\partial U}{\partial t}\right)_M + 4\pi r^2 \frac{\partial\left(F\rho^\gamma\right)}{\partial M} = 0. \tag{11}$$

Let us pass in (9) - (11) to new desired functions

$$R = r^3, \quad C = r^2 U.$$

After the pass the equations (9) - (11) take the form

$$\left(\frac{\partial R}{\partial t}\right)_M - 3C = 0, \tag{12}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_M + 4\pi\rho^2 \frac{\partial C}{\partial M} = 0, \tag{13}$$

$$\left(\frac{\partial C}{\partial t}\right)_M + 4\pi R^{\frac{4}{3}} \frac{\partial\left(F\rho^\gamma\right)}{\partial M} - 2C^2 R^{-1} = 0. \tag{14}$$

New functions on the shock wave have the form

$$R_w = R_0 \frac{M_w}{M_0}, \quad C_w = C_0 \left(\frac{M_w}{M_0}\right)^{(n-1)/n} \tag{15}$$

Let us proceed from the variables *t, M* to variables *t, ξ (t, M)*.
We define the dependence *ξ (t, M)* in the form

$$\xi = \frac{M}{M_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{-n}. \tag{16}$$

With the help of the equations for the derivatives

$$\left(\frac{\partial}{\partial t}\right)_M = \left(\frac{\partial}{\partial t}\right)_\xi + \left(\frac{\partial}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial t}\right)_M, \qquad \left(\frac{\partial}{\partial M}\right)_t = \left(\frac{\partial}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial M}\right)_t$$

where

$$\frac{\partial \xi}{\partial t} = \frac{n\xi}{t_f - t}, \qquad \frac{\partial \xi}{\partial M} = \frac{1}{M_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{-n},$$

we transform the equations (12)–(14)

$$\left(\frac{\partial R}{\partial t}\right)_{\xi}+\left(\frac{\partial R}{\partial \xi}\right)_{t}\left(\frac{\partial \xi}{\partial t}\right)_{M}-3C=0, \tag{17}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{\xi}+\left(\frac{\partial \rho}{\partial \xi}\right)_{t}\left(\frac{\partial \xi}{\partial t}\right)_{M}+4\pi\rho^{2}\left(\frac{\partial C}{\partial \xi}\right)_{t}\left(\frac{\partial \xi}{\partial M}\right)_{t}=0, \tag{18}$$

$$\left(\frac{\partial C}{\partial t}\right)_{\xi}+\left(\frac{\partial C}{\partial \xi}\right)_{t}\left(\frac{\partial \xi}{\partial t}\right)_{M}-\frac{2C^{2}}{R}+ \tag{19}$$

$$+4\pi R^{\frac{4}{3}}\left[\rho^{\gamma}\left(\frac{\partial F}{\partial \xi}\right)_{t}\left(\frac{\partial \xi}{\partial M}\right)_{t}+\gamma F\rho^{\gamma-1}\left(\frac{\partial \rho}{\partial \xi}\right)_{t}\left(\frac{\partial \xi}{\partial M}\right)_{t}\right]=0.$$

To separate the variables representing R, ρ and C in the form of products of functions of the time and the function of ξ

$$R = \alpha_R(t)T(\xi) \quad \rho = \alpha_\rho(t)\delta(\xi) \quad C = \alpha_C(t)Z(\xi)$$

We obtain these dependencies on the shock wave at $\xi=1$ with the help (14) and (15)

$$R_w = R_0 \cdot \left(\frac{t_f - t}{t_f - t_0}\right)^n, \quad C_w = C_0 \cdot \left(\frac{t_f - t}{t_f - t_0}\right)^{n-1}. \tag{20}$$

Then

$$T_w = T_1, \quad \alpha_R(t) = R_0 \left(\frac{t_f - t}{t_f - t_0}\right)^n T_1^{-1},$$

$$\delta_w = \delta_1, \quad \alpha_\rho = \rho_0 \left(\frac{\gamma+1}{\gamma-1}\right)\delta_1^{-1}, \tag{21}$$

$$Z_w = Z_1, \quad \alpha_C(t) = C_0 \left(\frac{t_f - t}{t_f - t_0}\right)^{n-1} Z_1^{-1}.$$

By substituting (20), (21) in (17)–(19), we obtain three equations for T, δ and Z

$$\xi T' = A_1, \tag{22}$$

$$\delta_1 B_1 Z' - \xi Z_1 \delta' = 0, \tag{23}$$

$$-\xi Z_1^{-1} Z' + C_1 \gamma \xi \delta_1^{-1} \delta' = C_2, \tag{24}$$

where

$$A_1 = T - \frac{2ZT_1}{(\gamma+1)Z_1}, \quad B_1 = \frac{2\delta^2}{(\gamma-1)\delta_1^2}, \quad C_1 = \frac{T^{4/3}\delta^{\gamma-1}\xi^{-(n+6)/3n}}{T_1^{4/3}\delta_1^{\gamma-1}},$$

$$C_2 = \frac{4Z^2T_1}{3(\gamma+1)Z_1^2 T} - \frac{(n-1)Z}{nZ_1} - C_1\frac{2(n-3)\delta}{3n\delta_1}.$$

The equations (22) - (24) are the system of linear inhomogeneous equations regarding to the T', δ', Z'.
The determinant of the system is the following

$$\Delta = B_1 C_1 \gamma \xi - \xi^2.$$

If $\Delta \neq 0$, the solution of the system (22)–(24) has the form

$$T' = \frac{A_1}{\xi}, \quad \delta' = \frac{B_1 C_2}{\Delta}, \quad Z' = \frac{\xi C_2}{\Delta}.$$

The values n_* and ξ_*, in which simultaneously $\Delta(\xi_*)=0$ and $C_2(\xi_*)=0$ corresponding to the values γ are given in the table

γ	1.1	1.2	4/3	1.4	5/3
n_*	2.387916	2.271434	2.183068	2.151532	2.065135
ξ_*	7.959997	5.717071	4.559431	4.227062	3.481885

In the area $n < n_*$ $\Delta(\xi)>0$ for $1 \le \xi < \infty$

In the area $n > n_*$ $\Delta(\xi_n)=0$, but $C_2(\xi_n)\neq 0$. On the border of the gas sphere at $M=M_0$ the value of ξ_n is reached at the moment

$$t_n = t_f - \left(t_f - t_0\right)\xi_n^{-1/n}.$$

From the point M_0 ,t_n comes out the line, on which $\xi=\xi_n$.

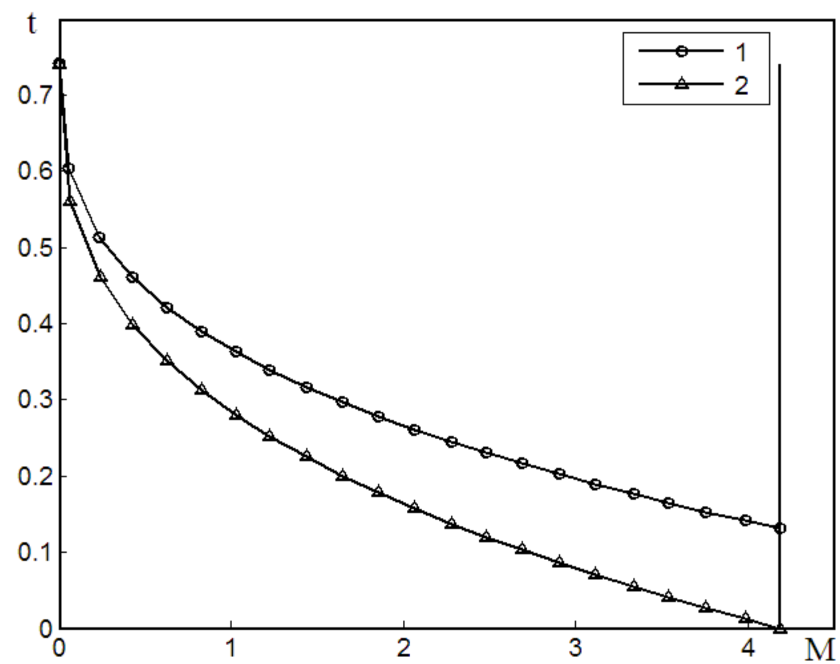
The equation of line is

$$M = M_0\xi_n\left(\frac{t_f - t}{t_f - t_0}\right)^n . \qquad (25)$$

This is a characteristic. This line is focuses together with the shock wave, because $M=0$ at $t=t_f$. In the area between (25) and the shock wave (8) for each $n>n_*$ the single solution is exists.

In Fig. 1 – It's a shock wave,

2 – It's a characteristic.



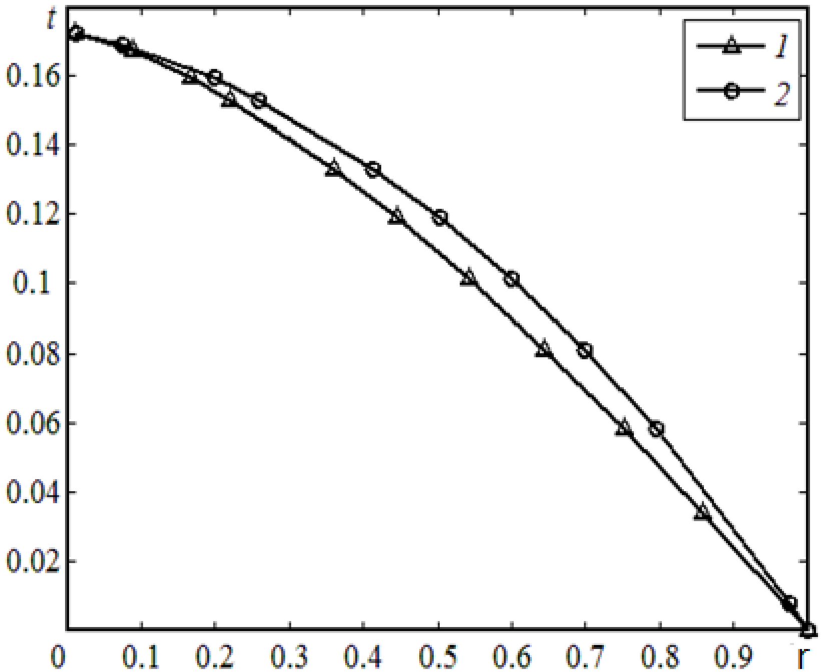


Fig. 1 The trajectories of the shock wave and the border of the gas sphere for γ=5/3 and n=0.68.

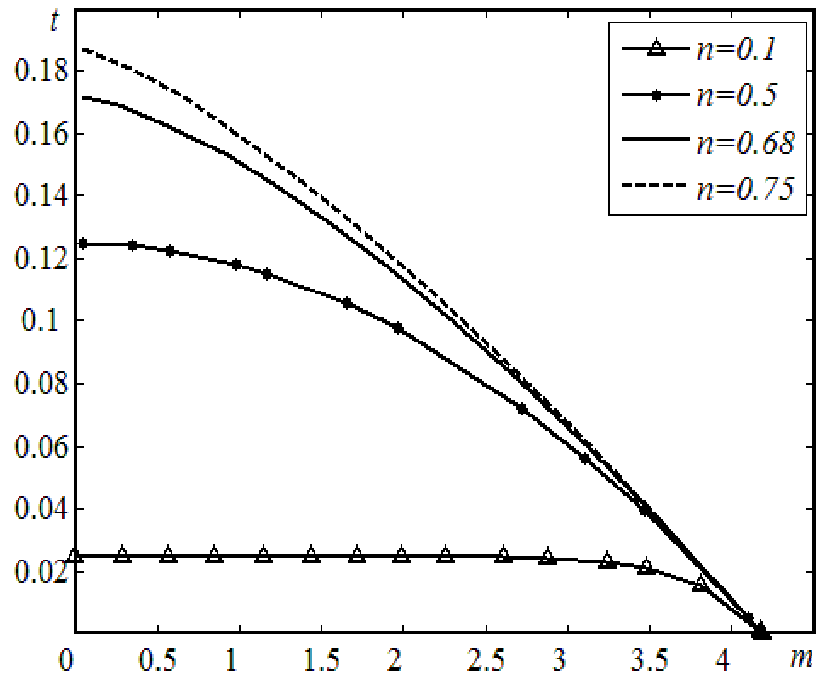
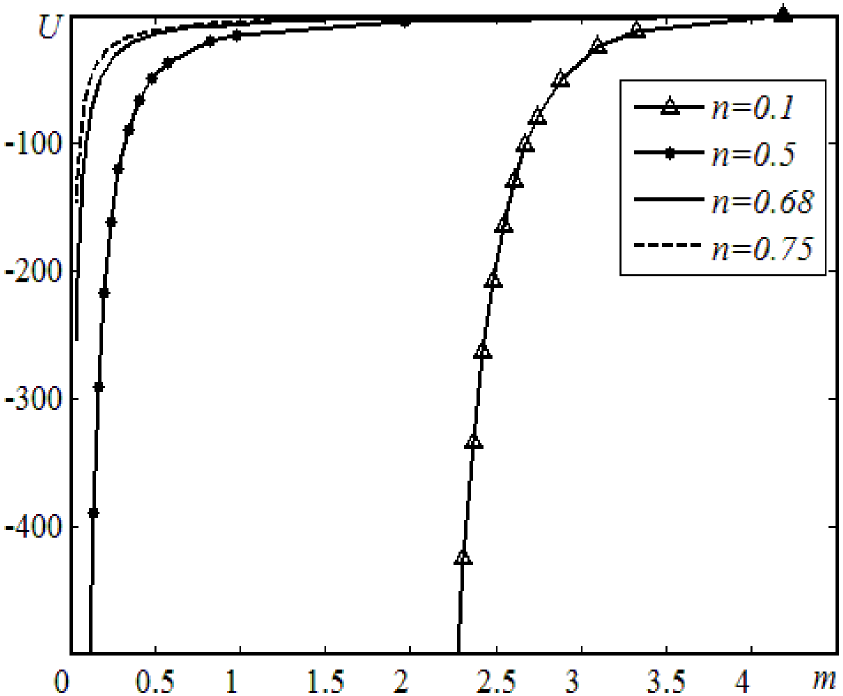


Fig. 2 The trajectories of the shock wave for γ=5/3 and four values n=0.1; 0.5; 0.68; 0.75 .



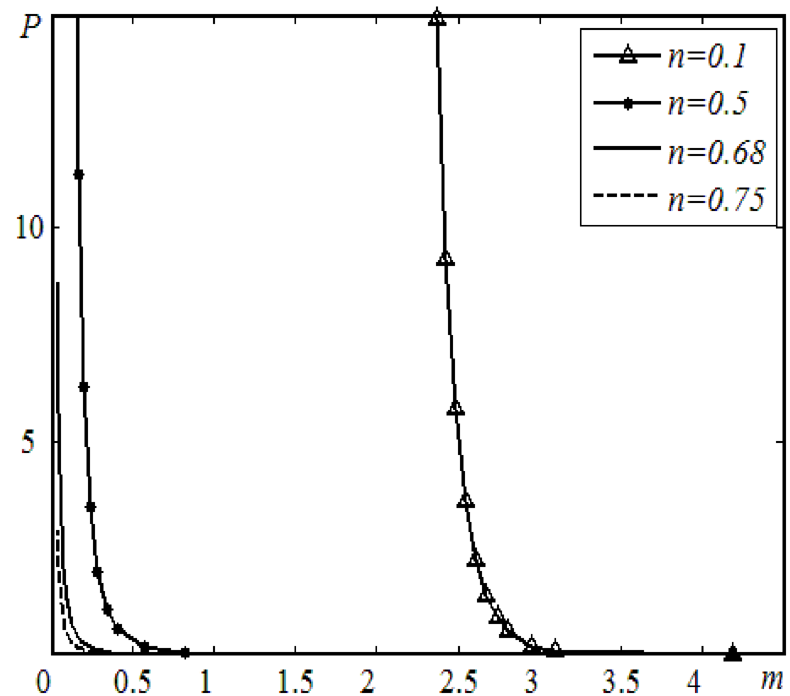
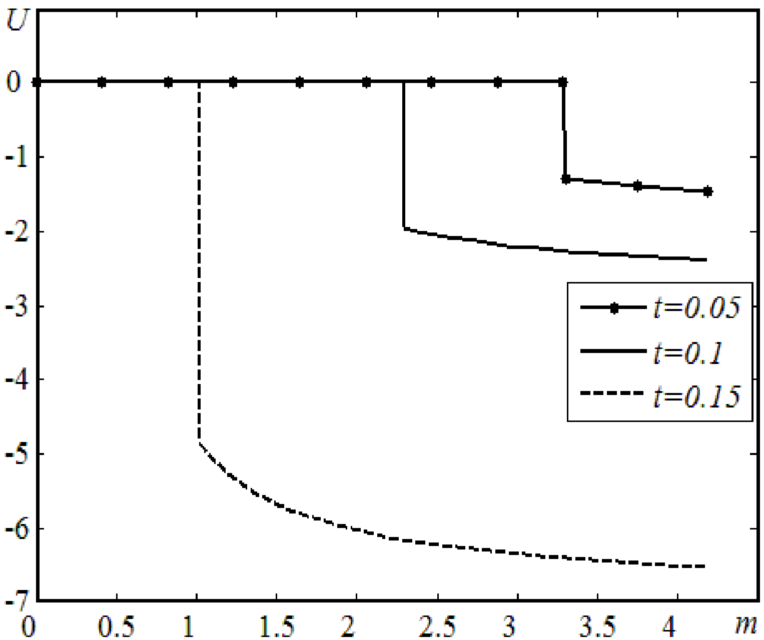
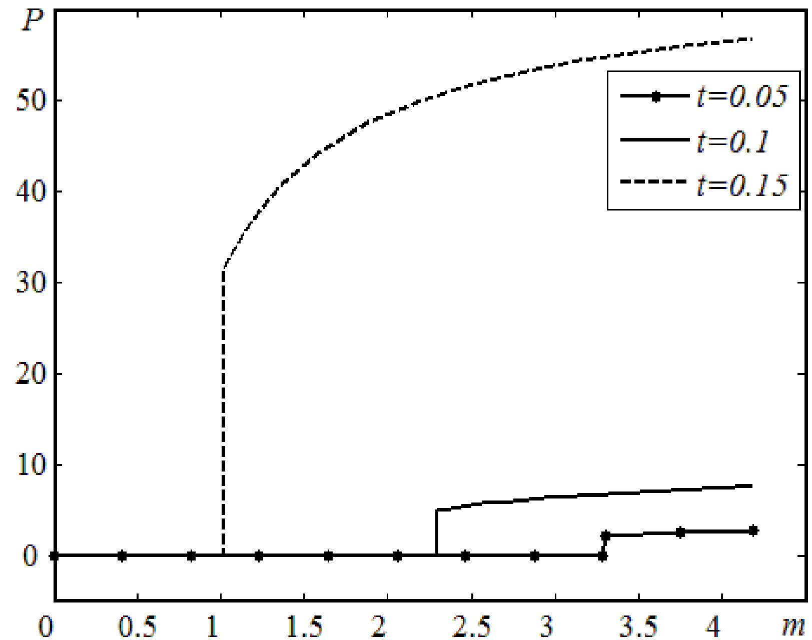


Fig. 3 Profiles of velocity on the shock wave for γ=5/3 and four values n=0.1; 0.5; 0.68; 0.75 in Lagrangian coordinates.

Fig. 4 Profiles of pressure on the shock wave for γ=5/3 and four values n=0.1; 0.5; 0.68; 0.75 in Lagrangian coordinates.





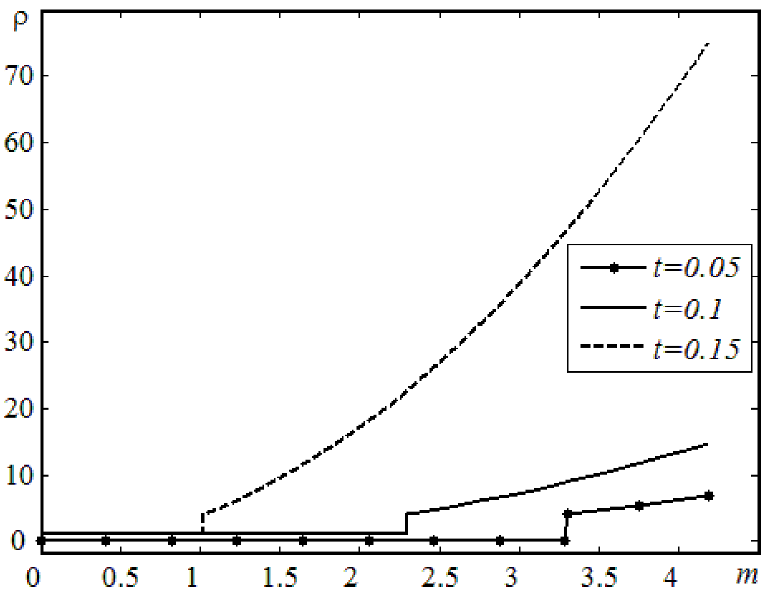
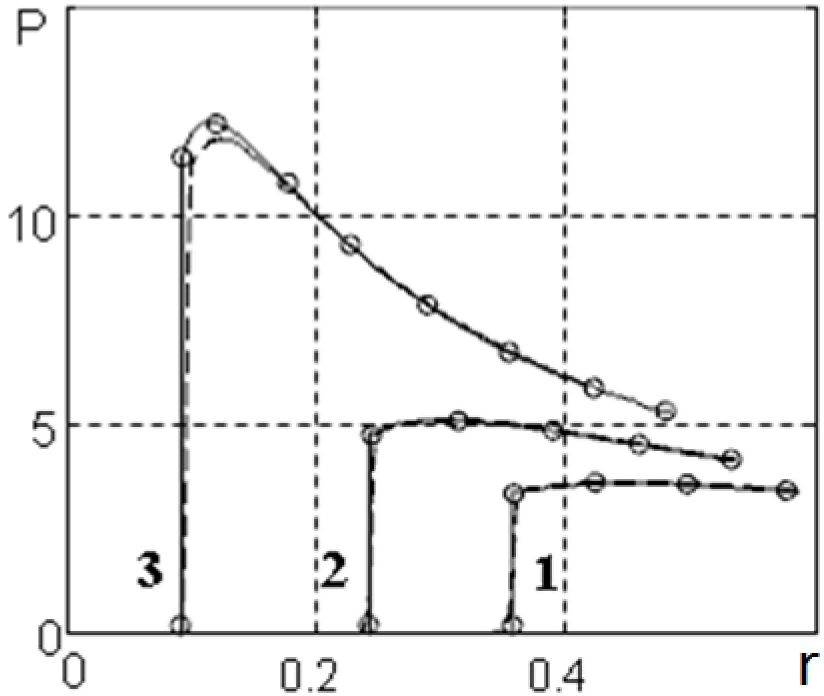
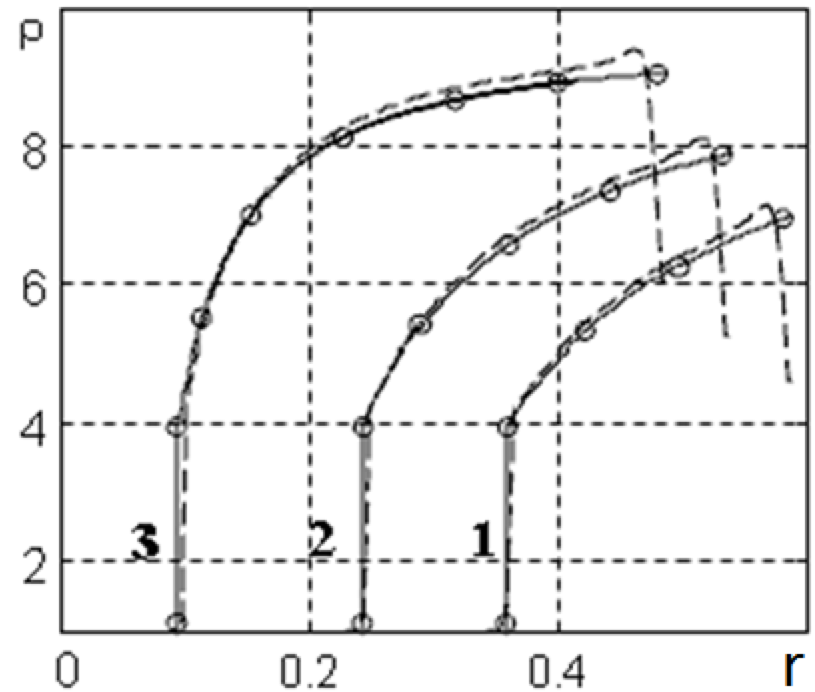


Fig. 5 Profiles of velocity, pressure and density for γ=5/3 and n=0.68 for time points t=0.05; 0.1; 0.15 in Lagrangian coordinates.





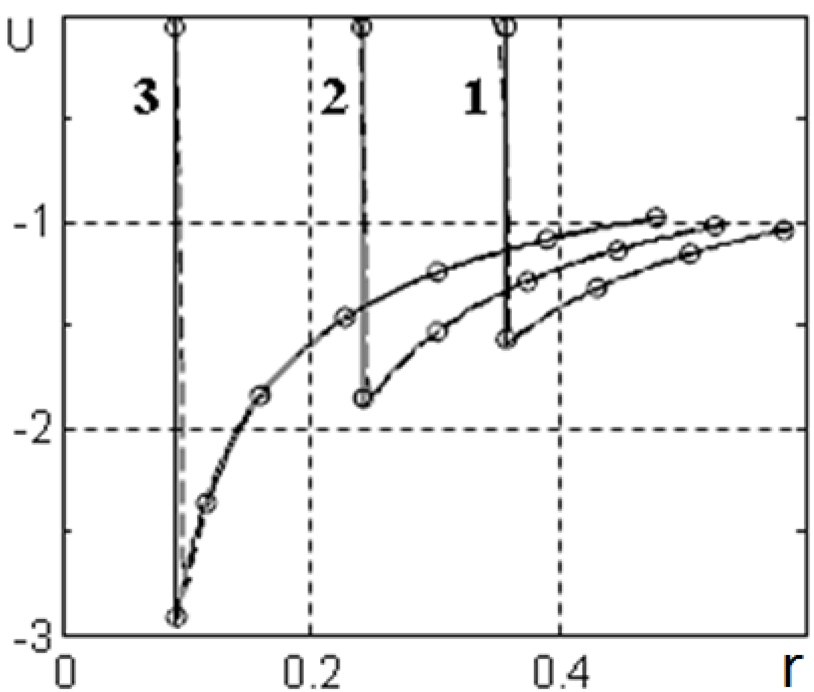
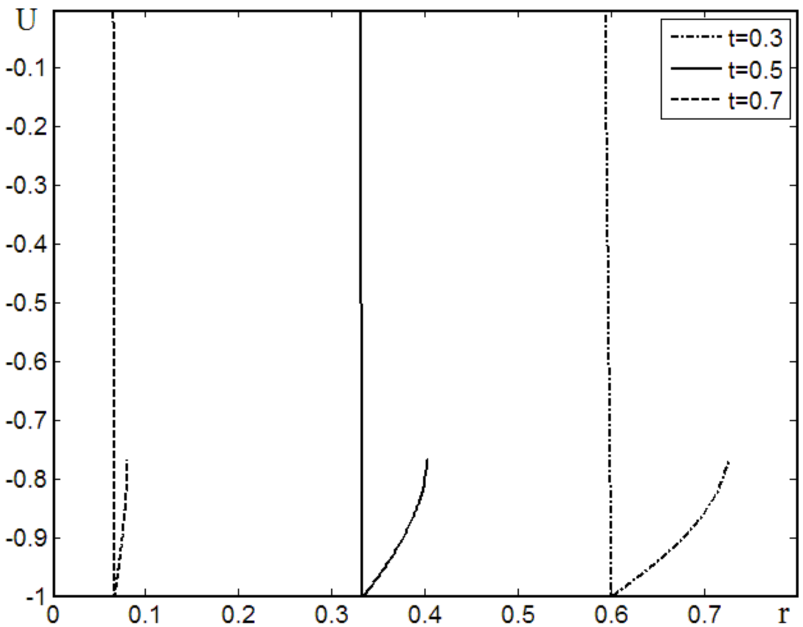


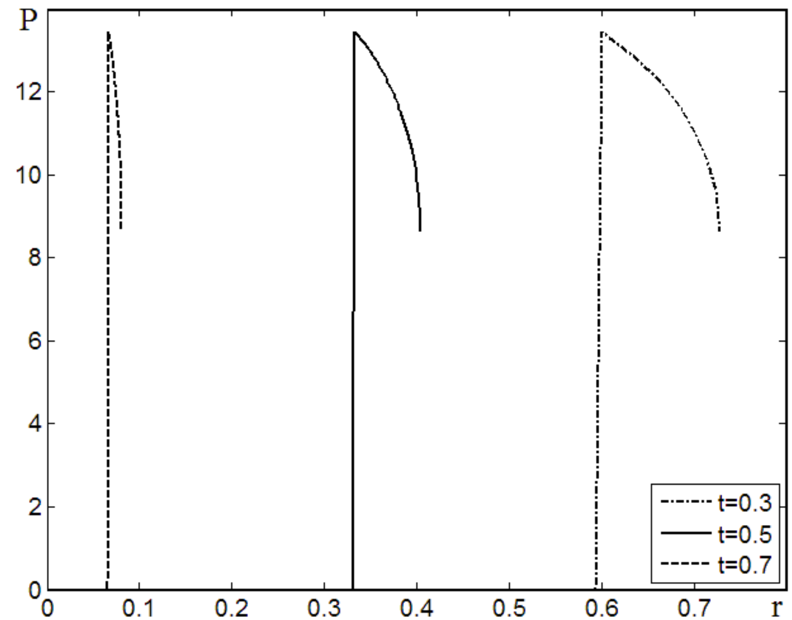
Fig. 6 Profiles of pressure, density and velocity for $\gamma=5/3$ and $n=n_*=2{,}065135$ for time points 1-t=0.4; 2-t=0.45; 3-t=0.5 in Euler coordinates.

The solid line – it's the analytical solution of this work,

-o- – the calculations on VOLNA program with the selection of discontinuity,

--- – calculations on VOLNA program without the selection of discontinuity.





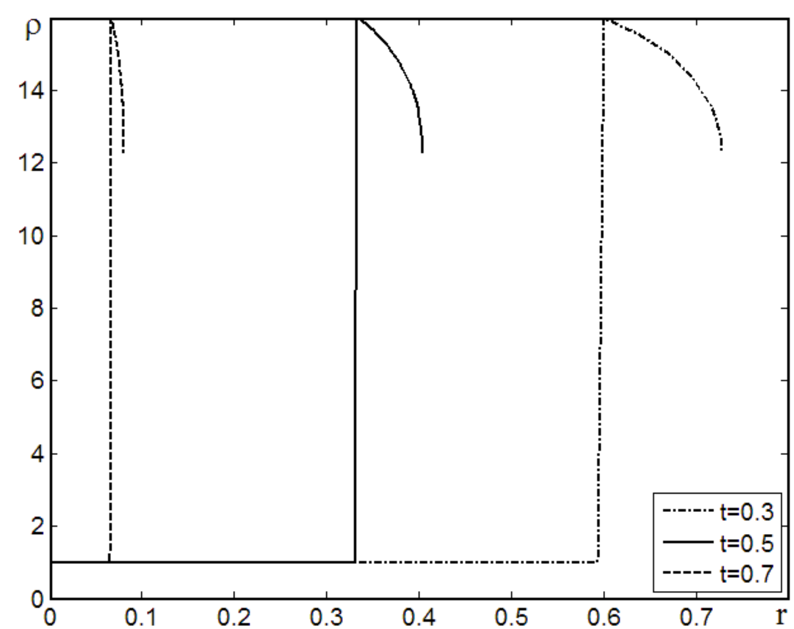


Fig. 7 Profiles of velocity, pressure and density between the shock wave and the characteristic for $\gamma=5/3$ and $n=3$ for time points $t=0.3$; 0.5; 0.7

in Euler coordinates.

Thank you for attention!